ON THE PROBLEM OF THE DIFFUSE REFLECTION OF LIGHT[1]

The problem of the diffuse reflection of light by a scattering medium consisting of plane-parallel layers is considered for multiple scattering.

The problem of the diffuse reflection of light by a medium, every volume element of which both absorbs and reflects (turbid medium), has remained unsolved, even for the simplest case of a medium consisting of plane-parallel layers and a parallel beam of rays falling on the boundary of the medium. The aim of the present paper is to show that in this case the problem admits a solution by rather simple means.

Consider a medium consisting of plane-parallel layers bounded on one side by a plane A and extending on the other side to infinity. A beam of parallel rays falls on the surface A and penetrates into the depths of the medium, undergoing absorption and diffusion. Denote by θ_0 the angle formed by the direction of the rays with the internal normal. Let φ_0 be the azimuth of the incident rays, computed from some given direction on the surface A.

The usual method of studying the problem consisted in considering the equation of flux:

$$\cos\theta \frac{\partial I(\tau,\theta,\varphi)}{\partial\tau} = I(\tau,\theta,\varphi) - B(\tau,\theta,\varphi) \tag{1}$$

and the generalized steady state equation for flux:

$$B(\tau,\theta,\varphi) = \\ = \frac{\lambda}{4\pi}\iint x(\cos\gamma)\, I(\tau,\theta',\varphi')\, d\omega' + \frac{\lambda}{4}\, S\, \exp\left(-\tau\sec\theta_0\right) x(\cos\gamma_1), \tag{2}$$

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which expresses the fact that the radiation of unit volume is made up of the energy of rays passing through the volume element in different directions, which is scattered by it, and of the scattered energy of the original beam attenuated $\exp(-\tau \sec\theta_0)$ times on its path to the given unit volume element. In these equations $I(\tau, \theta, \varphi)$ is the intensity of the diffused radiation at the optical depth τ in the direction forming an angle θ with the external normal and having an azimuth φ. The optical depth τ is determined for the usual linear depth z computed from the boundary A in terms of the volume coefficient of extinction of light $\alpha(z)$ by means of the formula:

$$\tau = \int_0^z \alpha(z)\, dz.$$

$B(\tau, \theta, \varphi)$ denotes the escaped radiation defined in terms of the coefficient of radiation $\eta(\tau, \theta, \varphi)$ at a depth τ in the direction θ, φ and α:

$$B(\tau, \theta, \varphi) = \frac{\eta(\tau, \theta, \varphi)}{\alpha}.$$

λ is the ratio of the coefficient of pure scattering to the sum of the coefficients of absorption and pure scattering; πS is the flow of external radiation falling on a unit surface perpendicular to it. And finally, $x(\cos\gamma)$ is a function called the scattering indicatrix, which gives the relative distribution of radiation scattered by a volume element from a direction θ, φ in a direction θ', φ' depending on the scattering angle γ between these two directions; the latter is defined by the formula:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'). \tag{3}$$

Instead of $x(\cos\gamma)$ we shall also write

$$x(\theta, \varphi; \theta', \varphi') = x(\cos\gamma).$$

As for the angle γ_1 between the direction θ, φ and the direction of the incident radiation,

$$\cos\gamma_1 = -\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0).$$

Equations (1) and (2) are usually solved approximately by averaging more or less roughly over the angles, or the system (1) and (2) is solved by reduction to one integral equation for an unknown function $B(\tau, \theta, \varphi)$ from which the intensity of radiation emerging from the medium is obtained by integrating. In our approach, the problem is reduced to a functional equation, which we shall proceed to solve.

Derivation of functional equation

The quantities I and B entering (1) depend not only on the arguments θ and φ, but also on parameters θ_0, φ_0 characterizing the direction of external radiation. In particular, the value of I for $\tau = 0$, i.e., the intensity of the diffuse radiation $I(0, \theta, \varphi)$ emerging from the boundary, which we call the diffusely reflected radiation, will also depend on parameters θ_0, φ_0. We now denote the intensity of this radiation by

$$I(0, \theta, \varphi) = r(\theta, \varphi; \theta_0, \varphi_0)\, S,$$

since due to the linear character of the problem it is proportional to the incident flow $\pi\, S$. The function $r(\theta, \varphi; \theta_0, \varphi_0)$ characterizes the diffuse reflection power of our medium. If the flux incident on the medium is not a beam of parallel rays, but radiation is coming from various directions θ_0, φ_0 where θ_0, as before, is the angle formed by the radiation with the internal normal, then the intensity of the diffusely reflected light $I_2(\theta, \varphi)$ will be equal to

$$I_2(\theta, \varphi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r(\theta, \varphi; \theta_0, \varphi_0)\, I_1(\theta_0, \varphi_0) \sin\theta_0 \, d\theta_0 \, d\varphi_0. \tag{4}$$

We shall look for the function r which we call the reflection function.

Let us return to the case of a parallel beam of rays and construct a plane A' at an optical depth $d\tau$ from the boundary plane A. At the plane A' we have two kinds of radiation penetrating the interior: 1) incident radiation $\pi\, S$, attenuated to the value $\pi\, S\,(1 - \sec\theta_0\, d\tau)$ and 2) radiation diffused from the above layer of optical depth $d\tau$. Its intensity, as evident from the equation of flux, equals

$$-B(0, \theta, \varphi)\ \sec\theta\, d\tau,$$

where $\sec\theta < 0$ since for flux moving towards the interior of the medium the angle with the external normal $\theta > \frac{\pi}{2}$. If instead of θ we introduce the angle formed with the internal normal $\theta' = \pi - \theta$, then the same intensity is given by

$$B(0, \pi - \theta', \varphi) \sec\theta' \, d\tau.$$

The part of the medium under A' reflects both radiations diffusely; the function r characterizing reflection remains the same, since the removal of a layer $d\tau$ from a medium of infinite optical thickness does not affect the diffuse reflection power of the medium. This invariance of the diffuse reflection power of a medium with respect to the removal or addition of a layer comprises the point of departure in our method.

Utilizing the definition of the function r we find that A' should reflect in the direction θ, φ with an intensity

$$S\, r(\theta, \varphi; \theta_0, \varphi_0)(1 - \sec\theta_0 \, d\tau) + \\ + \frac{d\tau}{\pi} \iint r(\theta, \varphi; \theta', \varphi') \, B(0, \pi - \theta', \varphi') \, \sec\theta' \sin\theta' \, d\theta' \, d\varphi'.$$

On the other hand, on the basis of the same equation of flux, we can write down the intensity of radiation moving outwards from the plane A'. Indeed, for $\tau = 0$ the intensity of radiation directed outwards equals $r(\theta, \varphi; \theta_0, \varphi_0)\, S$. Hence, according to the equation of flux, at a depth $d\tau$ it will be

$$S\, r(\theta, \varphi; \theta_0, \varphi_0)(1 + \sec\theta \, d\tau) - B(0, \theta, \varphi) \sec\theta \, d\tau.$$

Equating these two expressions for the intensity of the outward flux from the plane A', we obtain:

$$(\sec\theta + \sec\theta_0) r(\theta, \varphi; \theta_0, \varphi_0)\, S = \\ = B(0, \theta, \varphi) \sec\theta + \frac{1}{\pi} \iint r(\theta, \varphi; \theta', \varphi') \, B(0, \pi - \theta', \varphi') \, \tan\theta' \, d\theta' \, d\varphi'. \tag{5}$$

On the other hand, putting $\tau = 0$ in (2), we find for $B(0, \theta, \varphi)$

$$B(0, \theta, \varphi) = \frac{\lambda}{4} \cdot S\, x(\cos\gamma_1) + \frac{\lambda S}{4\pi} \iint x(\cos\gamma) \, I(0, \theta', \varphi') \sin\theta' \, d\theta' \, d\varphi'$$

or, since

$$I(0, \theta', \varphi') = S\, r(\theta', \varphi'; \theta_0, \varphi_0),$$

we have

$$B(0,\theta,\varphi) = \frac{\lambda}{4} \cdot S\,x(\cos\gamma_1) + \\ + \frac{\lambda S}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} x(\theta,\varphi;\theta',\varphi')\, r(\theta',\varphi';\theta_0,\varphi_0) \sin\theta'\, d\theta'\, d\varphi'. \tag{6}$$

Inserting (6) in (5) we find

$$(\sec\theta + \sec\theta_0) r(\theta,\varphi;\theta_0,\varphi_0) = \frac{\lambda}{4} \cdot x(\cos\gamma_1) \sec\theta + \\ + \frac{\lambda}{4\pi} \sec\theta \int_0^{2\pi} \int_0^{\frac{\pi}{2}} x(\theta,\varphi;\theta',\varphi')\, r(\theta',\varphi';\theta_0,\varphi_0) \sin\theta'\, d\theta'\, d\varphi' + \\ + \frac{\lambda}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r(\theta,\varphi;\theta',\varphi')\, x(\theta',\varphi';\theta_0,\varphi_0)\, \tan\theta'\, d\theta'\, d\varphi' + \\ + \frac{\lambda^2}{4\pi^2} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r(\theta,\varphi;\theta',\varphi')\, x(\pi-\theta',\varphi';\theta'',\varphi'') \times \\ \times r(\theta'',\varphi'';\theta_0,\varphi_0) \tan\theta' \sin\theta''\, d\theta'\, d\varphi'\, d\theta''\, d\varphi''. \tag{7}$$

This is the main functional equation for the function $r(\theta,\varphi;\theta_0,\varphi_0)$ characterizing diffuse reflection. For convenience in writing we shall take the argument of the functions r and x to be $\eta = \cos\theta$. We shall write $\cos\theta' = \eta'$, $\cos\theta'' = \eta''$ also. In this case our functional equation can be rewritten in the form

$$\left(\frac{1}{\eta} + \frac{1}{\eta_0}\right) r(\eta,\varphi;\eta_0,\varphi_0) = \frac{\lambda}{4\eta} \cdot x(\eta,\varphi;-\eta_0,\varphi_0) + \\ + \frac{\lambda}{4\pi\eta} \int_0^{2\pi} \int_0^{1} x(\eta,\varphi;\eta',\varphi')\, r(\eta',\varphi';\eta_0,\varphi_0)\, d\eta'\, d\varphi' + \\ + \frac{\lambda}{4\pi} \int_0^{2\pi} \int_0^{1} r(\eta,\varphi;\eta',\varphi')\, x(\eta',\varphi';\eta_0,\varphi_0)\, \frac{d\eta'}{\eta'}\, d\varphi' + \\ + \frac{\lambda}{4\pi^2} \int_0^{2\pi} \int_0^{1} \int_0^{2\pi} \int_0^{1} r(\eta,\varphi;\eta',\varphi')\, x(-\eta',\varphi';\eta'',\varphi'') \times \\ \times r(\eta'',\varphi'';\eta_0,\varphi_0)\, \frac{d\eta'}{\eta'}\, d\varphi'\, d\eta''\, d\varphi''. \tag{8}$$

This equation becomes more symmetrical if we insert

$$r(\eta,\varphi;\eta_0,\varphi_0) = \frac{\lambda}{4\eta} R(\eta,\varphi;\eta_0,\varphi_0). \tag{9}$$

Indeed, we shall then have:

$$\begin{aligned}
&\left(\frac{1}{\eta}+\frac{1}{\eta_0}\right) R(\eta,\varphi;\eta_0,\varphi_0) = x(\eta,\varphi;-\eta_0,\varphi_0)+\\
&+\frac{\lambda}{4\pi}\int_0^{2\pi}\int_0^1 x(\eta,\varphi;\eta',\varphi')\,R(\eta',\varphi';\eta_0,\varphi_0)\,\frac{d\eta'}{\eta'}\,d\varphi'+\\
&+\frac{\lambda}{4\pi}\int_0^{2\pi}\int_0^1 R(\eta,\varphi;\eta',\varphi')\,x(\eta',\varphi';\eta_0,\varphi_0)\,\frac{d\eta'}{\eta'}\,d\varphi'+\\
&+\frac{\lambda^2}{16\pi^2}\int_0^{2\pi}\int_0^1\int_0^{2\pi}\int_0^1 R(\eta,\varphi;\eta',\varphi')\,x(-\eta',\varphi';\eta'',\varphi'')\times\\
&\times R(\eta'',\varphi'';\eta_0,\varphi_0)\,\frac{d\eta'}{\eta'}\,d\varphi'\frac{d\eta''}{\eta''}\,d\varphi''.
\end{aligned} \tag{10}$$

This equation possesses the following property: if it is satisfied by some function $R(\eta,\varphi;\eta_0,\varphi_0)$, then it is also satisfied by the function $R(\eta_0,\varphi_0;\eta,\varphi)$ (arguments reversed). On the other hand, since our physical problem has but one solution, equation (10) also has but one regular solution. Hence,

$$R(\eta,\varphi;\eta_0,\varphi_0) = R(\eta_0,\varphi_0;\eta,\varphi), \tag{11}$$

i.e., the function R is symmetrical. This is in complete agreement with the symmetry of the expression

$$\eta\, r(\eta,\varphi;\eta_0,\varphi_0) = \frac{\lambda}{4}\, R(\eta,\varphi;\eta_0,\varphi_0)$$

derived from physical considerations in [1].

Expression of the indicatrix by means of Legendre polynomials

In several theoretical approaches (Rayleigh's formula), the scattering indicatrix is represented in the form of a finite sum of Legendre polynomials. In the general case it can be developed in a series by Legendre polynomials. If we assume only $n+1$ terms, then

$$x(\cos\gamma) = \sum_{i=0}^{n} x_i\, P_i(\cos\gamma). \tag{12}$$

Since the function $x(\cos\gamma)$ gives the relative distribution of the directions of scattered light in an elementary act of scattering, it satisfies the normalization condition

$$\frac{1}{4\pi}\iint x(\cos\gamma)\,d\omega = 1$$

or

$$\frac{1}{2}\int_0^{\pi} x(\cos\gamma)\sin\gamma\,d\gamma = \frac{1}{2}\int_{-1}^{+1} x(\eta)\,d\eta = 1. \tag{13}$$

Condition (13) gives the value $x_0 = 1$ for the first coefficient in the series (12).

In the case of a spherical scattering indicatrix we have simply

$$x(\cos\gamma) = 1,$$

i.e., only one term in the development.

In the case of the Rayleigh scattering indicatrix ,

$$x(\cos\gamma) = \frac{3}{4}(1+\cos^2\gamma) =$$
$$1 + \frac{1}{2}\left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right) = P_0(\cos\gamma) + \frac{1}{2}P_2(\cos\gamma),$$

i.e., in this case $x_0 = 1$ and $x_2 = 1/2$, while all the other coefficients are equal to zero.

Of considerable interest is the group of elongated scattering indicatrices of the type

$$x(\eta) = 1 + x_1\eta,$$

where the quantity x_1 characterizes the degree of elongation of the indicatrix in the direction of the incident ray.

In what follows we shall utilize the development (12) to solve the main functional equation (10), assuming that n can be both finite and infinite.

Solution of the main functional equation

According to the well-known formula for the addition of spherical functions:

$$P_i(\cos\gamma) = P_i\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\right) = P_i(\cos\theta)P_i(\cos\theta') +$$
$$+\,2\sum_{m=1}^{i}\frac{(i-m)!}{(i+m)!}P_i^m(\cos\theta)P_i^m(\cos\theta')\cos[m(\varphi-\varphi')].$$

Hence the scattering indicatrix $x(\cos\gamma)$ can be written in the form

$$x(\cos\gamma) = \sum_{m=0}^{\infty}\sum_{i=m}^{\infty} c_{im}\, P_i^m(\cos\theta)\, P_i^m(\cos\theta')\, \cos[m(\varphi-\varphi')] = \\ = \sum_{m=0}^{\infty}\sum_{i=m}^{\infty} c_{im}\, P_i^m(\eta)\, P_i^m(\eta')\, \cos[m(\varphi-\varphi')], \tag{14}$$

where

$$c_{im} = (2-\delta_{0m})\, x_i \frac{(i-m)!}{(i+m)!} \qquad (m = 0, 1, 2, \ldots), \tag{14'}$$

and

$$\delta_{00} = 1 \quad \text{and} \quad \delta_{0m} = 0 \quad \text{if} \quad m \neq 0$$

or in the form

$$x(\cos\gamma) = \sum_{m=0}^{\infty} q_m(\eta,\eta')\cos[m(\varphi-\varphi')], \tag{15}$$

where

$$q_m(\eta,\eta') = \sum_{i=m}^{\infty} c_{im}\, P_i^m(\eta)\, P_i^m(\eta') \tag{15'}$$

are symmetrical functions of their arguments.

The reflection function r, and hence R, depend on the difference between the azimuths $\varphi-\varphi_0$ of the incident and reflected light.

Also R must be an even function of $\varphi-\varphi_0$, since it is invariant with respect to a change in the zero direction of the azimuths. In view of this, in the Fourier development of the function R will have the form

$$R(\eta,\varphi;\eta_0,\varphi_0) = \sum_{m=0}^{\infty} f_m(\eta,\eta_0)\cos[m(\varphi-\varphi_0)]. \tag{16}$$

Our task consists in finding the functions $f_m(\eta,\eta_0)$, i.e., the Fourier coefficients of the function R. We insert (16) and (15) in (10). This gives

$$\left(\frac{1}{\eta}+\frac{1}{\eta_0}\right)\sum_{m=0}^{\infty} f_m(\eta,\eta_0)\cos[m(\varphi-\varphi_0)] =$$

$$\sum_{m=0}^{\infty} q_m(\eta,-\eta_0)\cos[m(\varphi-\varphi_0)]+$$

$$+\frac{\lambda}{2}\sum_{m=0}^{\infty}\frac{\cos[m(\varphi-\varphi_0)]}{2-\delta_{0m}}\int_0^1 q_m(\eta,\eta')\,f_m(\eta',\eta_0)\frac{d\eta'}{\eta'}+$$

$$+\frac{\lambda}{2}\sum_{m=0}^{\infty}\frac{\cos[m(\varphi-\varphi_0)]}{2-\delta_{0m}}\int_0^1 q_m(\eta',\eta_0)\,f_m(\eta,\eta')\frac{d\eta'}{\eta'}+$$

$$+\frac{\lambda^2}{4}\sum_{m=0}^{\infty}\frac{\cos[m(\varphi-\varphi_0)]}{(2-\delta_{0m})^2}\int_0^1\int_0^1 f_m(\eta,\eta')q_m(-\eta',\eta'')f_m(\eta'',\eta_0)\frac{d\eta'}{\eta'}\frac{d\eta''}{\eta''}.$$

Equating the coefficients of $\cos[m(\varphi-\varphi_0)]$ on both sides we find

$$\begin{aligned}\left(\frac{1}{\eta}+\frac{1}{\eta_0}\right) f_m(\eta,\eta_0) &= q_m(\eta-\eta_0)+\\ &+\frac{\lambda}{2(2-\delta_{0m})}\int_0^1 [q_m(\eta,\eta')\,f_m(\eta',\eta_0)+q_m(\eta',\eta_0)\,f_m(\eta,\eta')]\,\frac{d\eta'}{\eta'}+\\ &+\frac{\lambda^2}{4(2-\delta_{0m})^2}\int_0^1\int_0^1 f_m(\eta,\eta')\,q_m(-\eta',\eta'')\,f_m(\eta'',\eta_0)\frac{d\eta'}{\eta'}\frac{d\eta''}{\eta''}.\end{aligned} \tag{17}$$

We thus have obtained functional equations for the Fourier coefficients $f_m(\eta,\eta')$. Each $f_m(\eta,\eta_0)$ can be found separately.

It is evident from formulas (14) and (15) that if the development of the scattering indicatrix by Legendre polynomials contains only a finite number of terms, n being the degree of the highest term in the development, then for $m > n$ all the $q_m(\eta,\eta_0)$ disappear. Equations (17) show that in this case for $m > n$ the f_m also disappear.

In the Fourier development (16) of the reflection function the number of terms is equal to the degree of the highest term in the development (12) of the scattering indicatrix by Legendre polynomials.

Let us now investigate equations (17) for the functions $f_m(\eta,\eta_0)$. To this end we insert (15) in (17), which yields

$$\left(\frac{1}{\eta}+\frac{1}{\eta_0}\right) f_m(\eta,\eta_0) = \sum_{i=m}^{\infty}(-1)^{i+m} c_{im} P_i^m(\eta)\, P_i^m(\eta_0) +$$
$$+\frac{\lambda}{2(2-\delta_{0m})}\sum_{i=m}^{\infty} c_{im} P_i^m(\eta)\int_0^1 P_i^m(\eta')\, f_m(\eta',\eta_0)\frac{d\eta'}{\eta'}+$$
$$+\frac{\lambda}{2(2-\delta_{0m})}\sum_{i=m}^{\infty} c_{im} P_i^m(\eta_0)\int_0^1 P_i^m(\eta')\, f_m(\eta,\eta')\frac{d\eta'}{\eta'}+$$
$$+\sum_{i=m}^{\infty}\frac{\lambda^2(-1)^{i+m}c_{im}}{4(2-\delta_{0m})^2}\int_0^1 P_i^m(\eta') f_m(\eta,\eta')\frac{d\eta'}{\eta'}\int_0^1 P_i^m(\eta'') f_m(\eta'',\eta_0)\frac{d\eta''}{\eta''}.$$

We see that the right-hand side can be represented as a sum of products:

$$\left(\frac{1}{\eta}+\frac{1}{\eta_0}\right) f_m(\eta,\eta_0) =$$
$$= \sum_{i=m}^{\infty}(-1)^{i+m} c_{im}\left[P_i^m(\eta)+\frac{(-1)^{i+m}\lambda}{2(2-\delta_{0m})}\int_0^1 f_m(\eta,\eta')P_i^m(\eta')\frac{d\eta'}{\eta'}\right]\times \tag{18}$$
$$\times\left[P_i^m(\eta_0)+\frac{(-1)^{i+m}\lambda}{2(2-\delta_{0m})}\int_0^1 f_m(\eta',\eta_0)\, P_i^m(\eta')\,\frac{d\eta'}{\eta'}\right].$$

In view of the symmetry of the function $f_m(\eta,\eta_0)$ the two factors in parentheses, entering each term of the sum in formula (18), represent the same functions, one depending on η, the other on η_0. In other words, (18) can be rewritten

$$\left(\frac{1}{\eta}+\frac{1}{\eta_0}\right) f_m(\eta,\eta_0) = \sum_{i=m}^{\infty}(-1)^{i+m} c_{im}\, \varphi_i^m(\eta)\, \varphi_i^m(\eta_0), \tag{19}$$

where

$$\varphi_i^m(\eta) = P_i^m(\eta)+\frac{(-1)^{i+m}\lambda}{2(2-\delta_{0m})}\int_0^1 f_m(\eta,\eta')\, P_i^m(\eta')\,\frac{d\eta'}{\eta'}. \tag{20}$$

It follows from (19) that the function $f_m(\eta,\eta_0)$ has the structure

$$f_m(\eta,\eta_0) = \sum_{i=m}^{\infty}(-1)^{i+m}\frac{c_{im}\varphi_i^m(\eta)\cdot\varphi_i^m(\eta_0)}{\frac{1}{\eta}+\frac{1}{\eta_0}}. \tag{21}$$

The equations for the auxiliary functions $\varphi_i^m(\eta)$ are obtained by inserting (21) into the right-hand side of (20):

$$\varphi_i^m(\eta) = P_i^m(\eta) + \\ + \frac{(-1)^{i+m}\,\lambda}{2(2-\delta_{0m})} \int_0^1 \sum_{k=m}^{\infty} \frac{(-1)^{k+m} c_{km}\, \varphi_k^m(\eta)\, \varphi_k^m(\eta')}{\frac{1}{\eta} + \frac{1}{\eta'}} P_i^m(\eta') \cdot \frac{d\eta'}{\eta'}$$

or (using (14′))

$$\varphi_i^m(\eta) = P_i^m(\eta) + \\ + \frac{\lambda}{2} \sum_{k=m}^{\infty} (-1)^{i+k}\, \eta\, \frac{(k-m)!}{(k+m)!} \int_0^1 \frac{x_k\, \varphi_k^m(\eta)\, \varphi_k^m(\eta')}{\eta + \eta'} P_i^m(\eta')\, d\eta'. \qquad (22)$$

Putting $i = m, m+1, \ldots$, we derive from (22) a system of functional equations for the unknown functions $\varphi_m^m(r), \varphi_{m+1}^m(r), \ldots$ For each m the number of unknown functions is equal to the number of equations. If n is the degree of the highest Legendre polynomial in (12), then the total number of unknown functions as well as the number of functional equations is $\frac{(n+1)(n+2)}{2}$. However, the entire group of equations divides into a number of subsystems, corresponding to various m. The subsystem for the function φ_i^m, where m has a fixed value, can be solved independently of the other subsystems.

We summarize: the unknown function R of four variables by means of formulas (16) and (21) is expressed via functions of one variable $\varphi_i^m(\eta)$, the latter functions themselves being determined by the system of functional equations (22).

This form of representing R is especially convenient in the case of a finite development (12) of the scattering indicatrix, as seen from the particular examples which follow.

Spherical scattering indicatrix

In this case

$$x(\cos\gamma) = 1 \qquad (23)$$

and all the x_i's are zero for $i > 0$. The highest polynomial in the development of the scattering indicatrix is of zero degree. Therefore, only one term is left in the development (16), corresponding to $m = 0$

$$R = f_0(\eta, \eta_0). \tag{24}$$

Formula (21) is reduced in this case to

$$f_0(\eta, \eta_0) = \frac{\varphi_0^0(\eta)\,\varphi_0^0(\eta_0)}{\frac{1}{\eta} + \frac{1}{\eta_0}}, \tag{25}$$

and the unique auxiliary function $\varphi_0^0(\eta)$ is determined from the one functional equation

$$\varphi_0^0(\eta) = 1 + \frac{\lambda}{2}\eta\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')\,d\eta'}{\eta + \eta'}, \tag{26}$$

to which system (22) is reduced. For each λ this system is easily solved numerically by the method of successive approximations.

The case of spherical scattering indicatrix has been discussed by us in [2] where tables for different values of λ obtained are given.

Elongated scattering indicatrix

Consider an indicatrix of the type

$$x(\cos\gamma) = 1 + x_1\cos\gamma. \tag{27}$$

If $x_1 > 0$ such an indicatrix is elongated in the direction $\gamma = 0$, whereas in the case of $x_1 < 0$ it is elongated in the direction $\gamma = \pi$.

In these cases the degree of the highest Legendre polynomials in the development of the indicatrix is one. Hence (16) reduces to

$$R(\eta, \varphi; \eta_0, \varphi_0) = f_0(\eta, \eta_0) + f_1(\eta, \eta_0)\cos(\varphi - \varphi_0). \tag{28}$$

As for f_0 and f_1, according to (21), they have the following structure:

$$\begin{aligned} f_0(\eta, \eta_0) &= \frac{\varphi_0^0(\eta)\,\varphi_0^0(\eta_0) - x_1\varphi_1^0(\eta)\,\varphi_1^0(\eta_0)}{\frac{1}{\eta} + \frac{1}{\eta_0}}, \\ f_1(\eta, \eta_0) &= x_1\frac{\varphi_1^1(\eta)\,\varphi_1^1(\eta_0)}{\frac{1}{\eta} + \frac{1}{\eta_0}}. \end{aligned} \tag{29}$$

The auxiliary functions φ_0^0 and φ_1^0 should be determined, according to (22), from the subsystems

$$\varphi_0^0(\eta) = 1 + \frac{\lambda}{2}\,\eta\,\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')\,d\eta'}{\eta+\eta'} - \frac{\lambda}{2}\,x_1\,\eta\,\varphi_1^0(\eta)\int_0^1 \frac{\varphi_1^0(\eta')\,d\eta'}{\eta+\eta'},$$

$$\varphi_1^0(\eta) = \tag{30}$$

$$= \eta - \frac{\lambda}{2}\,\eta\,\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')\,\eta'\,d\eta'}{\eta+\eta'} + \frac{\lambda}{2}\,x_1\,\eta\,\varphi_1^0(\eta)\int_0^1 \frac{\varphi_1^0(\eta')\,\eta'\,d\eta'}{\eta+\eta'},$$

and the auxiliary function $\varphi_1^1(\eta)$ from the equation

$$\varphi_1^1(\eta) = \sqrt{1-\eta^2} + \frac{\lambda}{4}\,x_1\,\eta\,\varphi_1^1(\eta)\int_0^1 \frac{\varphi_1^1(\eta')\sqrt{1-(\eta')^2}}{\eta+\eta'}\,d\eta'. \tag{31}$$

The last equation can be solved numerically by successive approximations. By multiplying by $\sqrt{1-\eta^2}$ and substituting $\varphi_1^1(\eta)\sqrt{1-\eta^2} = \psi(\eta)$ we bring it to the convenient form

$$\psi(\eta) = 1 - \eta^2 + \frac{\lambda\,x_1}{4}\eta\,\psi(\eta)\int_0^1 \frac{\psi(\eta')\,d\eta'}{\eta+\eta'}. \tag{32}$$

A simple relation exists between the functions φ_0^0 and φ_0^1 given by (30). To establish it we transform the second equation of this system by substituting

$$\frac{\eta'}{\eta+\eta'} = 1 - \frac{\eta}{\eta+\eta'}.$$

Hence

$$\varphi_1^1(\eta) = 1 - \frac{\lambda}{2}\cdot\eta\,\varphi_0^0(\eta)\int_0^1 \varphi_0^0(\eta')\,d\eta' + \frac{\lambda\,x_1}{2}\cdot\eta\,\varphi_1^0(\eta)\int_0^1 \varphi_1^0(\eta')\,d\eta' +$$
$$+ \frac{\lambda}{2}\eta^2\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')\,d\eta'}{\eta+\eta'} - \frac{\lambda\,x_1}{2}\eta^2\varphi_1^0(\eta)\int_0^1 \frac{\varphi_1^0(\eta')\,d\eta'}{\eta+\eta'}.$$

On the basis of the first equation in (30), the last two terms here can be replaced by

$$\eta\left(\varphi_0^0(\eta) - 1\right).$$

Therefore, in terms of constants:

$$\alpha = \int_0^1 \varphi_0^0(\eta)\,d\eta, \qquad \beta = \int_0^1 \varphi_1^0(\eta)\,d\eta, \tag{33}$$

we have

$$\varphi_1^0(\eta) = \frac{\left(1 - \frac{\lambda}{2} \cdot \alpha\right) \eta \, \varphi_0^0(\eta)}{1 - \frac{\lambda x_1}{2} \cdot \beta\eta}. \tag{34}$$

Table 1. Values of $\varphi_0^0(\eta)$.
Scattering indicatrix $x(\cos\gamma) = 1 + \cos\gamma$

$\eta \backslash \lambda$	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.054	1.070	1.088	1.109	1.134	1.166	1.248
0.2	1.080	1.106	1.136	1.171	1.215	1.276	1.450
0.3	1.097	1.130	1.168	1.216	1.276	1.365	1.642
0.4	1.108	1.146	1.192	1.249	1.324	1.439	1.829
0.5	1.115	1.157	1.208	1.274	1.382	1.502	2.013
0.6	1.119	1.163	1.219	1.291	1.391	1.535	2.194
0.7	1.120	1.164	1.226	1.304	1.414	1.599	2.375
0.8	1.120	1.167	1.236	1.312	1.431	1.637	2.552
0.9	1.118	1.166	1.230	1.317	1.443	1.669	2.730
1.0	1.115	1.164	1.228	1.318	1.452	1.696	2.909

Table 2. Values of $\varphi_1^0(\eta)$.
Scattering indicatrix $x(\cos\gamma) = 1 + \cos\gamma$

$\eta \backslash \lambda$	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.083	0.077	0.071	0.064	0.054	0.041	0.000
0.2	0.172	0.162	0.150	0.136	0.118	0.090	0.000
0.3	0.264	0.251	0.235	0.215	0.189	0.177	0.000
0.4	0.359	0.344	0.324	0.299	0.265	0.210	0.000
0.5	0.456	0.439	0.417	0.387	0.346	0.277	0.000
0.6	0.555	0.536	0.512	0.478	0.431	0.349	0.000
0.7	0.654	0.635	0.609	0.572	0.519	0.425	0.000
0.8	0.755	0.734	0.708	0.669	0.610	0.505	0.000
0.9	0.856	0.836	0.808	0.767	0.704	0.588	0.000
1.0	0.959	0.939	0.910	0.867	0.800	0.674	0.000

Thus $\varphi_1^0(\eta)$ is expressed through $\varphi_0^0(\eta)$, but two constants α and β enter, which are determined from (33). The first equation in (30) in conjunction with (34) and (33) completely determines the functions φ_0^0 and

φ_1^0. The numerical values of the functions φ_0^0, φ_1^0 and φ_1^1 obtained by the method of successive approximations for different λ and for $x_1 = 1$ are given in Tables 1, 2, and 3. Our solutions are accurate up to the third decimal.

Table 3. Values of $\varphi_1^1(\eta)$.
Scattering indicatrix $x(\cos\gamma) = 1 + \cos\gamma$

$\eta\backslash\lambda$	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.016	1.021	1.027	1.032	1.038	1.044	1.051
0.2	1.010	1.018	1.026	1.034	1.043	1.053	1.062
0.3	0.988	0.998	1.007	1.018	1.028	1.040	1.050
0.4	0.954	0.964	0.975	0.986	0.997	1.009	1.022
0.5	0.903	0.915	0.926	0.938	0.951	0.982	0.976
0.6	0.836	0.647	0.859	0.870	0.883	0.895	0.908
0.7	0.749	0.759	0.770	0.780	0.790	0.804	0.815
0.8	0.632	0.640	0.648	0.658	0.668	0.679	0.688
0.9	0.459	0.466	0.4730	0.479	0.486	0.494	0.502
1.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000

For an indicatrix of the type considered, special attention is attached to the case $\lambda = 1$ (i.e., pure scattering, in the absence of absorption) with arbitrary x_1. In this case the system (30) is satisfied if we put φ_1^0 and φ_0^0 equal to the solution of (26) for a spherical scattering indicatrix.

If we put $\lambda = 1$ and $\varphi_1^0 = 0$ in (30), then our two equations are reduced to the following:

$$\varphi_0^0(\eta) = 1 + \frac{1}{2}\cdot\eta\,\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')\,d\eta'}{\eta+\eta'}, \tag{35}$$

$$1 = \frac{1}{2}\cdot\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')\,\eta'\,d\eta'}{\eta+\eta'}. \tag{36}$$

The first of these is identical to (26). Let us prove that the second is also equivalent to (26).

Under the integral sign in the second equation we substitute

$$\frac{\eta'}{\eta+\eta'} = 1 - \frac{\eta}{\eta+\eta'}$$

and obtain

$$1 = \frac{1}{2}\varphi_0^0(\eta) \int_0^1 \varphi_0^0(\eta)\, d\eta' - \frac{1}{2}\eta\varphi_0^0(\eta) \int_0^1 \frac{\varphi_0^0(\eta')\, d\eta'}{\eta + \eta'}. \tag{37}$$

On the other hand, integrating (36) with respect to η we obtain:

$$1 = \frac{1}{2}\int_0^1 \int_0^1 \frac{\varphi_0^0(\eta)\, \varphi_0^0(\eta')\, \eta'\, d\eta'\, d\eta}{\eta + \eta'} =$$
$$= \frac{1}{4}\int_0^1 \int_0^1 \varphi_0^0(\eta)\, \varphi_0^0(\eta')\, d\eta\, d\eta' = \frac{1}{4}\left[\int_0^1 \varphi_0^0(\eta)\, d\eta\right]^2,$$

i.e.,

$$\int_0^1 \varphi_0^0(\eta)\, d\eta = 2.$$

Substituting this value in (37) we see that (36) reduces to (26). Thus, for $\lambda = 1$ both equations (30) are satisfied if $\varphi_0^0(\eta)$ is taken as equal to the solution of (26), and φ_1^0 equal to zero.

Therefore, on the basis of (28) and (29) we can write for the function of reflection R:

$$R = \frac{\varphi_0^0(\eta)\, \varphi_0^0(\eta_0)}{\frac{1}{\eta} + \frac{1}{\eta_0}} + x_1 \cdot \frac{\varphi_1^1(\eta)\, \varphi_1^1(\eta_0)}{\frac{1}{\eta} + \frac{1}{\eta_0}} \cdot \cos(\varphi - \varphi_0).$$

After averaging over the azimuth, the second term disappears and we come to the following remarkable result.

In the case of pure scattering the reflection function for an elongated indicatrix $x(\cos\gamma) = 1 + x_1 \cos\gamma$, averaged over the azimuth, is exactly equal to the reflection function (25) for a spherical scattering indicatrix. For $\lambda \neq 1$ this rule no longer holds.

Remark on Lambert's law

In photometry in addition to the reflection function $r(\eta, \varphi; \eta_0, \varphi_0)$, use is often made of the coefficient of brightness

$$\rho = \frac{r}{\eta} = \frac{\lambda}{4\eta\eta_0} R.$$

According to Lambert's empirical law for certain material in which λ is close to unity, ρ is constant. In the case of a spherical scattering indicatrix it follows from (25) that

$$\rho = \frac{\lambda}{4} \frac{\varphi_0^0(\eta)\, \varphi_0^0(\eta_0)}{\eta + \eta'}.$$

For $\lambda = 1$, using the values of $\varphi_0^0(\eta)$, we present in Table 4 the values of the coefficient of brightness for various pairs of values of η and η_0. We see that for angles θ and θ_0, not too close to 90°, the quantity ρ is almost constant and has values close to unity. On the basis of the results of the preceding paragraph, we can assert that for a spherical indicatrix of the type $1 + x_1 \cos\gamma$ the mean (with respect to the azimuth) coefficient of brightness is almost constant and close to unity. For an arbitrary scattering indicatrix and for $\lambda = 1$, the coefficient of brightness, averaged over the azimuth, for not too large angles with the normal, is almost constant and close to unity.

Table 4. Spherical scattering indicatrix.
Coefficients of brightness for $\lambda = 1.0$

$\eta \backslash \eta$	−0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	3.12	1.810	1.370	1.140	1.010	0.913	0.849	0.798	0.758	0.728
0.1	3.12	1.95	1.510	1.280	1.140	1.050	0.977	0.926	0.884	0.862	0.825
0.2	1.81	1.51	1.320	1.190	1.110	1.040	0.994	0.957	0.925	0.900	0.878
0.3	1.37	1.28	1.190	1.130	1.070	1.030	1.000	0.975	0.953	0.946	0.918
0.4	1.14	1.14	1.110	1.070	1.050	1.020	1.000	0.987	0.972	0.961	0.951
0.5	1.01	1.05	1.040	1.030	1.020	1.010	1.000	0.997	0.988	0.981	0.977
0.6	0.913	0.977	0.994	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.998
0.7	0.849	0.926	0.957	0.975	0.987	0.997	1.000	1.010	1.010	1.010	1.020
0.8	0.798	0.884	0.925	0.953	0.972	0.988	1.000	1.010	1.020	1.020	1.030
0.9	0.759	0.852	0.900	0.946	0.961	0.981	0.998	1.010	1.020	1.030	1.040
1.0	0.728	0.825	0.878	0.918	0.951	0.977	0.998	1.020	1.030	1.040	1.060

Hence, the theory brings us to the following conclusions and limitations in the applicability of Lambert's law:

1) It is applicable only to purely scattering media.

2) It refers only to a coefficient of brightness averaged over the azimuth and does not take into account the dependence on the azimuth.

3) Even with these limitations it is valid only for not too large angles of incidence and reflection (up to 70°).

This method of solving the classical problem of the scattering of light can be easily generalized for the case of layers of finite optical thickness.

REFERENCES

1. Minnaert, Astrophys. Journ., 93, 403 (1941).
2. Ambartsumian, Astronom. Journ., 19, No. 1 (1942).

30 September 1943 Leningrad State University